Vector

1. Vectors & Their Representation:

Vector quantities are specified by definite magnitude and definite directions. A vector is generally represented by a directed line segment, say $\overrightarrow{AB}$. A is called the **initial point** & B is called the **terminal point**. The magnitude of vector $\overrightarrow{AB}$ is expressed by $|\overrightarrow{AB}|$.

Zero Vector: A vector of zero magnitude is a zero vector. i.e. which has the same initial & terminal point, is called a **Zero Vector.** It is denoted by **O**. The direction of zero vector is indeterminate.

Unit Vector: A vector of unit magnitude in the direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically, $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$.

Equal Vectors: Two vectors are said to be equal if they have the same magnitude, direction & represent the same physical quantity.

Collinear Vectors: Two vectors are said to be collinear if their directed line segments are parallel irrespective of their directions. Collinear vectors are also called **parallel vectors**. If they have the same direction they are named as **like vectors** otherwise **unlike vectors**.

Symbolically, two non zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\vec{a} = K\vec{b}$, where $K \in R$

Vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are collinear if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$

Coplanar Vectors: A given number of vectors are called coplanar if their line segments are all parallel to the same plane. Note that **"Two Vectors Are Always Coplanar"**.

Solved Example Find unit vector of $\hat{i} - 2\hat{j} + 3\hat{k}$

Solution $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ if $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

then $|\vec{a}| = \sqrt{a_x^{\,2} + a_y^{\,2} + a_z^{\,2}}$ $\therefore$ $|\vec{a}| = \sqrt{14}$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{14}}\,\hat{i} - \frac{2}{14}\,\hat{j} + \frac{3}{\sqrt{14}}\,\hat{k}$$

Solved Example Find values of x & y for which the vectors $\vec{a} = (x + 2)\,\hat{i} - (x - y)\,\hat{j} + \hat{k}$

$\vec{b} = (x - 1)\,\hat{i} + (2x + y)\,\hat{j} + 2\hat{k}$ are parallel.

Solution $\vec{a}$ and $\vec{b}$ are parallel if $\dfrac{x+2}{x-1} = \dfrac{y-x}{2x+y} = \dfrac{1}{2}$

$x = -5,\ y = -20$

2. Angle Between two Vectors

It is the smaller angle formed when the initial points or the terminal points of the two vectors are brought together. It should be noted that $0^\circ \le \theta \le 180^\circ$.

3. Addition Of Vectors:

- If two vectors $\vec{a}$ & $\vec{b}$ are represented by $\overrightarrow{OA}$ & $\overrightarrow{OB}$, then their sum $\vec{a} + \vec{b}$ is a vector represented by $\overrightarrow{OC}$, where OC is the diagonal of the parallelogram OACB.
- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutative)
- $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (associativity)
- $\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}$
- $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$
- $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$
- $|\vec{a} - \vec{b}| \ge ||\vec{a}| - |\vec{b}||$
- $\left|\vec{a} \pm \vec{b}\right| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 \pm 2|\vec{a}||\vec{b}|\cos\theta}$ where θ is the angle between the vectors
- A vector in the direction of the bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\dfrac{\vec{a}}{|\vec{a}|} + \dfrac{\vec{b}}{|\vec{b}|}$. Hence bisector of the angle between the two vectors $\vec{a}$ and $\vec{b}$ is $\lambda\left(\hat{a} + \hat{b}\right)$, where $\lambda \in R^+$. Bisector of the exterior angle between $\vec{a}$ & $\vec{b}$ is $\lambda\left(\hat{a} - \hat{b}\right)$, $\lambda \in R^+$.

4. Multiplication Of A Vector By A Scalar:

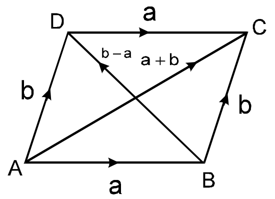
If $\vec{a}$ is a vector & m is a scalar, then m $\vec{a}$ is a vector parallel to $\vec{a}$ whose modulus is $|m|$ times that of $\vec{a}$. This multiplication is called **Scalar Multiplication**. If $\vec{a}$ and $\vec{b}$ are vectors & m, n are scalars, then:

$m(\vec{a}) = (\vec{a})\, m = m\vec{a}$ $\qquad$ $m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

$(m+n)\,\vec{a} = m\vec{a} + n\vec{a}$ $\qquad$ $m(a+\vec{b}) = m\vec{a} + m\vec{b}$

Solved Example: If $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ represent two adjacent sides of a parallelogram, find unit vectors parallel to the diagonals of the parallelogram.

Solution. Let ABCD be a parallelogram such that $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$.



Then, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \Rightarrow \overrightarrow{AC} = \vec{a} + \vec{b} = 3\hat{i} + 6\hat{j} - 2\hat{k}$

and $\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD} \Rightarrow \overrightarrow{AD} + \overrightarrow{AD} = \overrightarrow{AB}$

$\Rightarrow \overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB} = \vec{b} - \vec{a}$

Now, $\overrightarrow{AC} = 3\hat{i} + 6\hat{j} - 2\hat{k} \Rightarrow |\overrightarrow{AC}| = \sqrt{9+36+4} = 7$

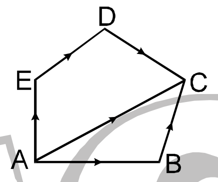
and, $\overrightarrow{BD} = \hat{i} + 2\hat{j} - 8\hat{k} \Rightarrow |\overrightarrow{BD}| = \sqrt{1+4+64} = \sqrt{69}$

$\therefore$ Unit vector along $\overrightarrow{AC} = \dfrac{\overrightarrow{AC}}{|\overrightarrow{AC}|} = \dfrac{1}{7}\left(3\hat{i} + 6\hat{j} - 2\hat{k}\right)$

and, Unit vector along $\overrightarrow{BD} = \dfrac{\overrightarrow{BD}}{|\overrightarrow{BD}|} = \dfrac{1}{\sqrt{69}}\left(\hat{i} + 2\hat{j} - 8\hat{k}\right)$

Solved Example ABCDE is a pentagon. Prove that the resultant of the forces $\overrightarrow{AB}$, $\overrightarrow{AE}$, $\overrightarrow{BC}$, $\overrightarrow{DC}$, $\overrightarrow{ED}$ and $\overrightarrow{AC}$ is $3\overrightarrow{AC}$.

Solution. Let R be the resultant force



$\therefore$ R = $\overrightarrow{AB} + \overrightarrow{AE} + \overrightarrow{BC} + \overrightarrow{DC} + \overrightarrow{ED} + \overrightarrow{AC}$

$\therefore$ R = $(\overrightarrow{AB} + \overrightarrow{BC}) + (\overrightarrow{AE} + \overrightarrow{ED} + \overrightarrow{DC}) + \overrightarrow{AC}$

$= \overrightarrow{AC} + \overrightarrow{AC} + \overrightarrow{AC}$

$= 3\overrightarrow{AC}$. Hence proved.

Self Practice Problems :

1. Express : (i) The vectors $\overrightarrow{BC}$ $\overrightarrow{CA}$ and $\overrightarrow{AB}$ in terms of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$

(ii) The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and in terms of the vectors $\overrightarrow{OC}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

Ans. (i) $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$, $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

2. Given a regular hexagon ABCDEF with centre O, show that

(i) $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OD}$ $\qquad$ (ii) $\overrightarrow{OD} + \overrightarrow{OA} = 2\,\overrightarrow{OB} + \overrightarrow{OF}$ $\quad$ (iii) $\overrightarrow{AD} + \overrightarrow{EB} + \overrightarrow{PC} = 4\,\overrightarrow{AB}$

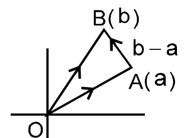
3. The vector $-\hat{i} + \hat{j} - \hat{k}$ bisects the angle between the vectors $\vec{c}$ and $3\hat{i} + 4\hat{j}$. Determine the unit vector along $\vec{c}$. **Ans.** $-\dfrac{1}{3}\hat{i} + \dfrac{2}{15}\hat{j} - \dfrac{14}{15}\hat{k}$

4. The sum of the two unit vectors is a unit vector. Show that the magnitude of the their difference is $\sqrt{3}$.

5. Position Vector Of A Point:

let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B, then,

$\overrightarrow{AB} = \vec{b} - \vec{a}$ = pv of B – pv of A.

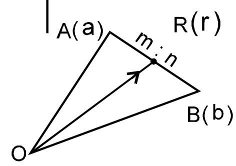


DISTANCE FORMULA

Distance between the two points $A(\vec{a})$ and $B(\vec{b})$ is AB = $\left|\vec{a} - \vec{b}\right|$

SECTION FORMULA

If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point which divides AB in the ratio m: n is given by: $\vec{r} = \dfrac{n\vec{a} + m\vec{b}}{m+n}$.



Note p.v. of mid point of AB = $\dfrac{\vec{a} + \vec{b}}{2}$.

Solved Example: ABCD is a parallelogram. If L, M be the middle point of BC and CD, express $\overrightarrow{AL}$ and $\overrightarrow{AM}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$, also show that $\overrightarrow{AL} + \overrightarrow{AM} = \dfrac{3}{2}\,\overrightarrow{AC}$.

Solution. Let the position vectors of points B and D be respectively $\vec{b}$ and $\vec{a}$ referred to A as origin of reference.

Then $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AD} + \overrightarrow{AB}$ $\qquad$ [$\because$ $\overrightarrow{DC} = \overrightarrow{AB}$]

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$= \vec{d} + \vec{b} \qquad \because \quad \overrightarrow{AB} = \vec{b},\ \overrightarrow{AD} = \vec{d}$

i.e. position vector of C referred to A is $\vec{d} + \vec{b}$

$\therefore\ \overrightarrow{AL}$ = p.v. of L, the mid point of $\overrightarrow{BC}$.

$= \frac{1}{2}$ [p.v. of D + p.v. of C] $= \frac{1}{2}\left(\vec{b} + \vec{d} + \vec{b}\right) = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD}$

$\overrightarrow{AM} = \frac{1}{2}\left[\vec{d} + \vec{d} + \vec{b}\right] = \overrightarrow{AD} + \frac{1}{2}\overrightarrow{AB}$

$\therefore\ \overrightarrow{AL} + \overrightarrow{AM} = \vec{b} + \frac{1}{2}\vec{d} + \vec{d} + \frac{1}{2}\vec{b}$

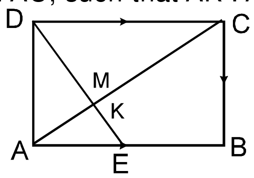
$= \frac{3}{2}\vec{b} + \frac{3}{2}\vec{d} \qquad = \frac{3}{2}(\vec{b} + \vec{d}) = \frac{3}{2}\overrightarrow{AC}$.

Solved Example If ABCD is a parallelogram and E is the mid point of AB, show by vector method that DE trisects and is trisected by AC.

Solution. Let $\overrightarrow{AB} = \vec{a}$ and AD = $\vec{b}$

Then $\overrightarrow{BC} = \overrightarrow{AD} = \vec{b}$ and $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD} = \vec{a} + \vec{b}$

Also let K be a point on AC, such that AK : AC = 1 : 3



or, $AK = \frac{1}{3}AC \Rightarrow \overrightarrow{AK} = \frac{1}{3}(\vec{a} + \vec{b})$ ....(i)

Again E being the mid point of AB, we have

$\overrightarrow{AE} = \frac{1}{2}\vec{a}$

Let M be the point on DE such that DM : ME = 2 : 1

$\therefore\ \overrightarrow{AM} = \frac{\overrightarrow{AD} + 2\overrightarrow{AE}}{1+2} = \frac{\vec{b} + \vec{a}}{3}$ ....(ii)

From (i) and (ii) we find that : $\overrightarrow{AK} = \frac{1}{3}(\vec{a} + \vec{b}) = \overrightarrow{AM}$, and so we conclude that K and M coincide. i.e. DE trisect AC and is trisected by AC. Hence proved.

Self Practice Problems

1. If $\vec{a}, \vec{b}$ are position vectors of the points (1, –1), (–2, m), find the value of m for which $\vec{a}$ and $\vec{b}$ are collinear. **Ans.** m = 2

2. The position vectors of the points A, B, C, D are $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 5\hat{j}$, $3\hat{i} + 2\hat{j} - 3\hat{k}$, $\hat{i} - 6\hat{j} - \hat{k}$ respectively. Show that the lines AB and CD are parallel and find the ratio of their lengths. **Ans.** 1 : 2

3. The vertices P, Q and S of a triangle PQS have position vectors $\vec{p}, \vec{q}$ and $\vec{s}$ respectively.

(i) Find $\vec{m}$, the position vector of M, the mid-point of PQ, in terms of $\vec{p}$ and $\vec{q}$.

(ii) Find $\vec{t}$, the position vector of T on SM such that ST : TM = 2 : 1, in terms of $\vec{p}, \vec{q}$ and $\vec{s}$.

(iii) If the parallelogram PQRS is now completed. Express $\vec{r}$, the position vector of the point R in terms of $\vec{p}, \vec{q}$ and $\vec{s}$

Prove that P, T and R are collinear.

Ans. $\vec{m} = \frac{1}{2}(\vec{p} + \vec{q})$, $\quad \vec{t} = \frac{1}{2}(\vec{p} + \vec{q} + \vec{s})$, $\quad \vec{r} = \frac{1}{2}\ \vec{q} + \vec{p} - \vec{s}$

4. D, E, F are the mid-points of the sides BC, CA, AB respectively of a triangle. Show $\overrightarrow{FE}$ = 1/2 $\overrightarrow{BC}$ and that the sum of the vectors $\overrightarrow{AD}$, $\overrightarrow{BE}$, $\overrightarrow{CF}$ is zero.

5. The median AD of a triangle ABC is bisected at E and BE is produced to meet the side AC in F; show that AF = 1/3 AC and EF = 1/4 BF.

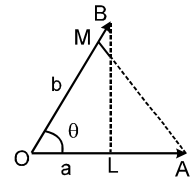
6. Point L, M, N divide the sides BC, CA, AB of ΔABC in the ratios 1 : 4, 3 : 2, 3 : 7 respectively. Prove that $\overrightarrow{AL} + \overrightarrow{BM} + \overrightarrow{CN}$ is a vector parallel to $\overrightarrow{CK}$, when K divides AB in the ratio 1 : 3.

6. Scalar Product Of Two Vectors:

Geometrical interpretation of Scalar Product

Let $\vec{a}$ and $\vec{b}$ be vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Draw BL $\perp$ OA and AM $\perp$ OB.

From Δs OBL and OAM, we have OL = OB cos θ and OM = OA cos θ. Here OL and OM are known as projections of $\vec{b}$ on $\vec{a}$ and $\vec{a}$ on $\vec{b}$ respectively.



Now, $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$

$= |\vec{a}|$ (OB cos θ)

$= |\vec{a}|$ (OL)

= (Magnitude of $\vec{a}$) (Projection of $\vec{b}$ on $\vec{a}$) ....(i)

Again $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = |\vec{b}|\,(|\vec{a}| \cos\theta)$

$= |\vec{b}|$ (OA cos θ)

$= |\vec{b}|$ (OM)

= (magnitude of $\vec{b}$) (Projection of $\vec{a}$ on $\vec{b}$) ....(ii)

Thus geometrically interpreted, the scalar product of two vectors is the product of modulus of either vector and the projection of the other in its direction.

1. i.i = j.j = k.k = 1; i.j = j.k = k.i = 0 ☞ projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

2. if $\vec{a} = a_1 i + a_2 j + a_3 k$ & $\vec{b} = b_1 i + b_2 j + b_3 k$ then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

3. the angle φ between $\vec{a}$ & $\vec{b}$ is given by $\cos\phi = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|}$ $0 \le \phi \le \pi$

4. $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta \ (0 \le \theta \le \pi)$, note that if θ is acute then $\vec{a}.\vec{b} > 0$ & if θ is obtuse then $\vec{a}.\vec{b} < 0$

5. $\vec{a}.\vec{a} = |\vec{a}|^2 = \vec{a}^2,\ \vec{a}.\vec{b} = \vec{b}.\vec{a}$ (commutative) ☞ $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ (distributive)

6. $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ $(\vec{a} \ne 0\ \ \vec{b} \ne 0)$

7. $(m\vec{a}).\vec{b} = \vec{a}.(m\vec{b}) = m(\vec{a}.\vec{b})$ (associative) where m is scalar.

Note : (i) Maximum value of $\vec{a}.\vec{b}$ is $|\vec{a}|\,|\vec{b}|$ **(ii)** Minimum value of $\vec{a}.\vec{b}$ is $-|\vec{a}|\,|\vec{b}|$

(iii) Any vector $\vec{a}$ can be written as, $\vec{a} = (\vec{a}.\hat{i})\,\hat{i} + (\vec{a}.\hat{j})\,\hat{j} + (\vec{a}.\hat{k})\,\hat{k}$.

Solved Example Find the value of p for which the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are

(i) perpendicular (ii) parallel

Solution. (i) $\vec{a} \perp \vec{b} \Rightarrow \vec{a}.\vec{b} = 0 \Rightarrow (3\hat{i} + 2\hat{j} + 9\hat{k}).(\hat{i} + p\hat{j} + 3\hat{k}) = 0$

$\Rightarrow 3 + 2p + 27 = 0 \Rightarrow p = -15$

(ii) We know that the vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are parallel iff

$\vec{a} = \lambda\vec{b} \Leftrightarrow (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = \lambda\,(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \Leftrightarrow a_1 = \lambda b_1,\ a_2 = \lambda b_2,\ a_3 = \lambda b_3$

$\Leftrightarrow \dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}\ (=\lambda)$

So, vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are parallel iff

$$\frac{3}{1} = \frac{2}{p} = \frac{9}{3} \Rightarrow 3 = \frac{2}{p} \Rightarrow p = 2/3$$

Solved Example: If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$.

Solution. We have, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \vec{a} + \vec{b} = -\vec{c} \Rightarrow (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = (-\vec{c}).(-\vec{c})$

$\Rightarrow |\vec{a} + \vec{b}|^2 = |\vec{c}|^2 \Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = |\vec{c}|^2$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\,|\vec{a}|\,|\vec{b}| \cos\theta = |\vec{c}|^2$

$\Rightarrow 9 + 25 + 2\,(3)\,(5) \cos\theta = 49 \Rightarrow \cos\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{3}$.

Solved Example Find the values of x for which the angle between the vectors $\vec{a} = 2x^2\,\hat{i} + 4x\,\hat{j} + \hat{k}$ and $\vec{b} = 7\,\hat{i} - 2\,\hat{j} + x\,\hat{k}$ is obtuse.

Solution. The angle q between vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|}$

Now, θ is obtuse $\Rightarrow$ $\cos\theta < 0$ $\Rightarrow$ $\dfrac{\vec{a}\,.\,\vec{b}}{|\vec{a}|\,|\vec{b}|} < 0$

$\Rightarrow$ $\vec{a}\,.\,\vec{b} < 0$ $[\because, |\vec{a}|, |\vec{b}| > 0]$

$\Rightarrow$ $14x^2 - 8x + x < 0$

$\Rightarrow$ $17x\,(2x - 1) < 0$ $\Rightarrow x(2x-1) < 0 \Rightarrow 0 < x < \dfrac{1}{2}$

Hence, the angle between the given vectors is obtuse if $x \in (0, 1/2)$

Solved Example: D is the mid point of the side BC of a triangle ABC, show that $AB^2 + AC^2 = 2\,(AD^2 + BD^2)$

Solution. We have

$\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB}$

$\Rightarrow$ $AB^2 = (\overrightarrow{AD} + \overrightarrow{DB})^2$

$= AD^2 + DB + 2\overrightarrow{AD}\,.\,\overrightarrow{DB}$ ....(i)

Also we have

$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$ $\Rightarrow$ $AC^2 = (\overrightarrow{AD} + \overrightarrow{DC})^2$

$= AD^2 + DC^2 + 2\overrightarrow{AD}\,.\,\overrightarrow{DC}$ ....(ii)

Adding (i) and (ii), we get

$AB^2 + AC^2 = 2AD^2 + 2BD^2 + 2\,\overrightarrow{AD}\,.\,(\overrightarrow{DB} + \overrightarrow{DC})$

$= 2(DA^2 + DB^2)$, for $\overrightarrow{DB} + \overrightarrow{DC} = 0$

Solved Example

If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$, then find

(i) Component of $\vec{b}$ along $\vec{a}$. (ii) Component of $\vec{b}$ perpendicular to along $\vec{a}$.

Solution. (i) Component of $\vec{b}$ along $\vec{a}$ is

$\left(\dfrac{\vec{a}\,.\,\vec{b}}{|\vec{a}|^2}\right)\vec{a}$

Here $\vec{a}\,.\,\vec{b} = 2 - 1 + 3 = 4$

$|\vec{a}|^2 = 3$

Hence $\left(\dfrac{\vec{a}\,.\,\vec{b}}{|\vec{a}|^2}\right)\vec{a} = \dfrac{4}{3}\vec{a} = \dfrac{4}{3}(\hat{i} + \hat{j} + \hat{k})$

(ii) Component of $\vec{b}$ perpendicular to along $\vec{a}$ is $\vec{b} - \left(\dfrac{\vec{a}\,.\,\vec{b}}{|\vec{a}|^2}\right)\vec{a}\,. = \dfrac{1}{3}\left(2\hat{i} - 7\hat{j} + 5\hat{k}\right)$

Self Practice Problems :1. If $\vec{a}$ and $\vec{b}$ are unit vectors and θ is angle between them, prove that $\tan\dfrac{\theta}{2} = \dfrac{|\vec{a}-\vec{b}|}{|\vec{a}+\vec{b}|}$.

2. Find the values of x and y is the vectors $\vec{a} = 3\hat{i} + x\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + y\hat{k}$ are mutually perpendicular vectors of equal magnitude. **Ans.** $x = -\dfrac{31}{12}$, $y = \dfrac{41}{12}$

3. Let $\vec{a} = x^2\hat{i} + 2\hat{j} - 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $c = x^2\hat{i} + 5\hat{j} - 4\hat{k}$ be three vectors. Find the values of x for which the angle between $\vec{a}$ and $\vec{b}$ is acute and the angle between $\vec{b}$ and $\vec{c}$ is obtuse. **Ans.** $(-3, -2) \cup (2, 3)$

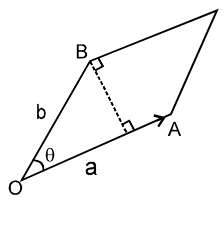
4. The points O, A, B, C, D, are such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = 2\vec{a} + 3\vec{b}$, $\overrightarrow{OD} = \vec{a} + 2\vec{b}$. Give that the length of $\overrightarrow{OA}$ is three times the length of $\overrightarrow{OB}$ show that $\overrightarrow{BD}$ and $\overrightarrow{AC}$ are perpendicular.

5. ABCD is a tetrahedron and G is the centroid of the base BCD. Prove that $AB^2 + AC^2 + AD^2 = GB^2 + GC^2 + GD^2 + 3GA^2$

7. Vector Product Of Two Vectors:

1. If $\vec{a}\,\&\,\vec{b}$ are two vectors & θ is the angle between them then $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\;\vec{n}$, where $\vec{n}$ is the unit vector perpendicular to both $\vec{a}\,\&\,\vec{b}$ such that $\vec{a}$, $\vec{b}$ & $\vec{n}$ forms a right handed screw system.

2. Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ & $\vec{b}$.



3. $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$; $\hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$

4. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

5. $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ (not commutative)

6. $(m\vec{a})\times\vec{b} = \vec{a}\times(m\vec{b}) = m(\vec{a}\times\vec{b})$ (associative) where m is a scalar.

7. $\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$ (distributive)

8. $\vec{a}\times\vec{b} = 0 \Leftrightarrow \vec{a}$ & $\vec{b}$ are parallel (collinear) $(\vec{a}\neq 0$, $\vec{b}\neq 0)$ i.e. $\vec{a} = K\vec{b}$, where K is a scalar.

9. Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\hat{n} = \pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

☞ A vector of magnitude 'r' & perpendicular to the palne of $\vec{a}$ & $\vec{b}$ is $\pm\dfrac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$

☞ If θ is the angle between $\vec{a}$ & $\vec{b}$ then $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

☞ If $\vec{a},\vec{b}$ & $\vec{c}$ are the pv's of 3 points A, B & C then the vector area of triangle ABC = $\dfrac{1}{2}\left[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}\right]$. The points A, B & C are collinear if $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = \vec{0}$

☞ Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\dfrac{1}{2}|\vec{d}_1\times\vec{d}_2|$

☞ Lagrange's Identity: for any two vectors $\vec{a}$ & $\vec{b}$; $(\vec{a}\times\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$

Solved Example

Find a vector of magnitude 9, which is perpendicular to both the vectors $4\hat{i}+\hat{j}+3\hat{k}$ and $-2\hat{i}+\hat{j}-2\hat{k}$.

Solution. Let $\vec{a} = 4\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b} = -2\hat{i}+\hat{j}-2\hat{k}$. Then,

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = (2-3)\ \hat{i} - (-8+6)\ \hat{j} + (4-2)\ \hat{k} = -\hat{i}+2\hat{j}+2\hat{k}$$

$\Rightarrow$ $|\vec{a}\times\vec{b}| = \sqrt{(-1)^2+2^2+2^2} = 3$

$\therefore$ Required vector = $9\left(\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}\right) = \dfrac{9}{3}(-\hat{i}+2\hat{j}+2\hat{k}) = -3\hat{i}+6\hat{j}+6\hat{k}$

Solved Example

For any three vectors $\vec{a},\vec{b},\vec{c}$. Show that $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b}) = \vec{0}$.

Solution. We have, $\vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}+\vec{b})$

$= \vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}+\vec{c}\times\vec{b}$ [Using distributive law]

$= \vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}-\vec{a}\times\vec{b}-\vec{a}\times\vec{c}-\vec{b}\times\vec{c}$ [$\because$ $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ etc]

Solved Example: For any vector $\vec{a}$, prove that $|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2 = 2|\vec{a}|^2$

Solution. Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$. Then

$\vec{a}\times\hat{i} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times\hat{i} = a_1(\hat{i}\times\hat{i}) + a_2(\hat{j}\times\hat{i}) + a_3(\hat{k}\times\hat{i}) = -a_2\hat{k}+a_3\hat{i}$

$\Rightarrow$ $|\vec{a}\times\hat{i}|^2 = a_2^2 + a_3^2$

$\vec{a}\times\hat{j} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times\hat{j} = a_1\hat{k}-a_3\hat{i}$

$\Rightarrow$ $|\vec{a}\times\hat{j}|^2 = a^2{}_1 + a_3^2$ $\Rightarrow$ $|\vec{a}\times\hat{k}|^2 = a_1^2 + a_2^2$

$\therefore$ $|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2 = a_2^2 + a_3^3 + a_1^2 + a_3^2 + a_1^2 + a_2^2$

$2(d_1^2 + a_2^2 + a_3^2) = 2|\vec{a}|^2$

Solved Example: Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = 10\vec{a} + 2\vec{b}$ and $\overrightarrow{OC} = \vec{b}$ where O is origin. Let p denote the area of the quadrilateral OABC and q denote the area of the parallelogram with OA and OC as adjacent sides. Prove that p = 6q.

Solution. We have,

p = Area of the quadrilateral OABC

$= \frac{1}{2}\,|\overrightarrow{OB}\times\overrightarrow{AC}|$

$= \frac{1}{2}\,|\overrightarrow{OB}\times(\overrightarrow{OC}-\overrightarrow{OA})|$

$= \frac{1}{2}\,|(10\vec{a}+2\vec{b})\times(\vec{b}-\vec{a})|$

$= \frac{1}{2}\,|10(\vec{a}\times\vec{b}-10(\vec{a}\times\vec{a})+2(\vec{b}\times\vec{b})-2(\vec{b}\times\vec{a})|$

$= \frac{1}{2}\,|10(\vec{a}\times\vec{b})-0+0+2(\vec{a}\times\vec{b})|$

and, q = Area of the parallelogram with OA and OC as adjacent sides

$= |\overrightarrow{OA}\times\overrightarrow{OC}| = |\vec{a}\times\vec{b}|$ ....(ii)

From (i) and (ii), we get p = 6q

Self Practice Problems :

1. If $\vec{p}$ and $\vec{q}$ are unit vectors forming an angle of 30º; find the area of the parallelogram having $\vec{a}=\vec{p}+2\vec{q}$ and $\vec{b}=2\vec{p}+\vec{q}$ as its diagonals. **Ans.** 3/4 sq. units

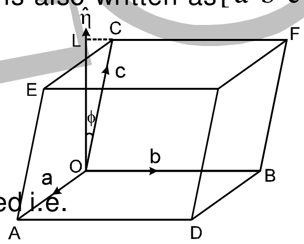
2. Show that $\{(\vec{a}+\vec{b}+\vec{c})\times(\vec{c}-\vec{b})\}\cdot\vec{a} = 2\,[\vec{a}\ \vec{b}\ \vec{c}]$.

3. Prove that the normal to the plane containing the three points whose position vectors are $\vec{a},\vec{b},\vec{c}$ lies in the direction $\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}$

4. ABC is a triangle and EF is any straight line parallel to BC meeting AC, AB in E, F respectively. If BR and CQ be drawn parallel to AC, AB respectively to meet EF in R and Q respectively, prove that Δ ARB = ΔACQ.

8. Scalar Triple Product:

☞ The scalar triple product of three vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ is defined as: $\vec{a}\times\vec{b}\cdot\vec{c} = |\vec{a}||\vec{b}||\vec{c}|\,\sin\theta\,\cos\phi$ where θ is the angle between $\vec{a}$ & $\vec{b}$ & ϕ is the angle between $\vec{a}\times\vec{b}$ & $\vec{c}$. It is also written as $[\vec{a}\ \vec{b}\ \vec{c}]$ and spelled as box product.



☞ Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by $\vec{a},\vec{b}\,\&\,\vec{c}$ i.e. $V=[\vec{a}\ \vec{b}\ \vec{c}]$

☞ In a scalar triple product the position of dot & cross can be interchanged i.e.

$\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}$ OR $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

☞ $\vec{a}\cdot(\vec{b}\times\vec{c}) = -\vec{a}\cdot(\vec{c}\times\vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

☞ If $\vec{a} = a_1i+a_2j+a_3k$; $\vec{b} = b_1i+b_2j+b_3k$ & $\vec{c} = c_1i+c_2j+c_3k$ then $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

In general, if $\vec{a} = a_1\vec{l} + a_2\vec{m} + a_3\vec{n}$; $\vec{b} = b_1\vec{l} + b_2\vec{m} + b_3\vec{n}$ & $\vec{c} = c_1\vec{l} + c_2\vec{m} + c_3\vec{n}$

then $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\left[\vec{l}\,\vec{m}\,\vec{n}\right]$; where $\vec{\ell}$, $\vec{m}$ & $\vec{n}$ are non coplanar vectors.

☞ If $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$.

☞ Scalar product of three vectors, two of which are equal or parallel is 0 i.e. $[\vec{a}\,\vec{b}\,\vec{c}] = 0$,

☞ If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non – coplanar then $[\vec{a}\ \vec{b}\ \vec{c}] > 0$ for right handed system & $[\vec{a}\ \vec{b}\ \vec{c}] < 0$ for left handed system.

☞ [i j k] = 1 ☞ $[K\vec{a}\ \vec{b}\ \vec{c}] = K[\vec{a}\ \vec{b}\ \vec{c}]$ ☞ $[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$

☞ The volume of the tetrahedron OABC with O as origin & the pv's of A, B and C being $\vec{a}$, $\vec{b}$ & $\vec{c}$ respectively is given by $V = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$

☞ The positon vector of the centroid of a tetrahedron if the pv's of its vertices are $\vec{a}$, $\vec{b}$, $\vec{c}$ & $\vec{d}$ are given by $\frac{1}{4}[\vec{a}+\vec{b}+\vec{c}+\vec{d}]$.

Note that this is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron.

Remember that: $\left[\vec{a}-\vec{b} \quad \vec{b}-\vec{c} \quad \vec{c}-\vec{a}\right] = 0$ & $\left[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}\right] = 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

Solved Example

Find the volume of a parallelopiped whose sides are given by $-3\hat{i}+7\hat{j}+5\hat{k}$, $-5\hat{i}+7\hat{j}-3\hat{k}$ and $7\hat{i}-5\hat{j}-3\hat{k}$

Solution. Let $\vec{a}=-3\hat{i}+7\hat{j}+5\hat{k}$, $\vec{b}=-5\hat{i}+7\hat{j}+3\hat{k}$ and $\vec{c}=7\hat{i}-5\hat{j}-3\hat{k}$.

We know that the volume of a parallelopiped whose three adjacent edges are $\vec{a},\vec{b},\vec{c}$ is $\left|[\vec{a},\vec{b},\vec{c}]\right|$.

Now, $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix} = -3(-21-15) - 7(15+21) + 5(25-49)$

$= 108 - 252 - 120 = -264$

So, required volume of the parallelopiped = $\left|[\vec{a},\vec{b},\vec{c}]\right|$ = | – 264 | = 264 cubic units

Solved Example: Simplify $[\vec{a}-\vec{b}\ \vec{b}-\vec{c}\ \vec{c}-\vec{a}]$

Solution. We have :

$[\vec{a}-\vec{b}\ \vec{b}-\vec{c}\ \vec{c}-\vec{a}] = \{(\vec{a}-\vec{b})\times(\vec{b}-\vec{c})\}\cdot(\vec{c}-\vec{a})$ [by def.]

$= (\vec{a}\times\vec{b}-\vec{a}\times\vec{c}-\vec{b}\times\vec{b}+\vec{b}\times\vec{c})\cdot(\vec{c}-\vec{a})$ [by dist. law]

$= (\vec{a}\times\vec{b}+\vec{c}\times\vec{a}+\vec{b}\times\vec{c})\cdot(\vec{c}-\vec{a})$ [$\because\ \vec{b}\times\vec{b}=0$]

$= (\vec{a}\times\vec{b})\cdot\vec{c} - (\vec{a}\times\vec{b})\cdot\vec{a} + (\vec{c}\times\vec{a})\cdot\vec{c} - (\vec{c}\times\vec{a})\cdot\vec{a} + (\vec{b}\times\vec{c})\cdot\vec{c} - (\vec{b}\times\vec{c})\cdot\vec{a}$ [by dist. law]

$= [\vec{a}\,\vec{b}\,\vec{c}] - [\vec{a}\,\vec{b}\,\vec{a}] + [\vec{c}\,\vec{a}\,\vec{c}] - [\vec{c}\,\vec{a}\,\vec{a}] + [\vec{b}\,\vec{c}\,\vec{c}] - [\vec{b}\,\vec{c}\,\vec{a}]$

$= [\vec{a}\,\vec{b}\,\vec{c}] - [\vec{b}\,\vec{c}\,\vec{a}]$ [$\because$ scalar triple product when any two vectors are equal is zero]

$= [\vec{a}\,\vec{b}\,\vec{c}] - [\vec{a}\,\vec{b}\,\vec{c}] = 0$ [$\because\ [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]$]

Solved Example: Find the volume of the tetrahedron whose four vertices have position vectors $\vec{a}$ $\vec{b}$ $\vec{c}$ and $\vec{d}$.

Solution. Let four vertices be A, B, C, D with p. v. $\vec{a}$ $\vec{b}$ $\vec{c}$ and $\vec{d}$. respectively.

$\therefore$ $\vec{DA} = (\vec{a}-\vec{d})$

$\vec{DB} = (\vec{b}-\vec{d})$

$\vec{DC} = (\vec{c}-\vec{d})$

Hence volume $= \frac{1}{6}[\vec{a}-\vec{d}\ \ \vec{b}-\vec{d}\ \ \vec{c}-\vec{d}]$

$= \frac{1}{6}(\vec{a}-\vec{d})\cdot[(\vec{b}-\vec{d})\times(\vec{c}-\vec{d})]$

$= \frac{1}{6}(\vec{a}-\vec{d})\cdot[\vec{b}\times\vec{c}-\vec{b}\times\vec{d}+\vec{c}\times\vec{d}]$

$= \frac{1}{6}\{[\vec{a}\ \vec{b}\ \vec{c}]-[\vec{a}\ \vec{b}\ \vec{d}]+[\vec{a}\ \vec{c}\ \vec{d}]-[\vec{d}\ \vec{b}\ \vec{c}]\}$

$= \frac{1}{6}\{[\vec{a}\ \vec{b}\ \vec{c}]-[\vec{a}\ \vec{b}\ \vec{d}]+[\vec{a}\ \vec{c}\ \vec{d}]-[\vec{b}\ \vec{c}\ \vec{d}]\}$

Solved Example: Show that the vectors $\vec{a}=-2\hat{i}+4\hat{j}-2\hat{k}$, $\vec{b}=4\hat{i}-2\hat{j}-2\hat{k}$ and $\vec{c}=-2\hat{i}-2\hat{j}+4\hat{k}$ are coplanar.

Solution: The vectors are coplanar since $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} -2 & 4 & -2 \\ 4 & -2 & -2 \\ -2 & -2 & 4 \end{vmatrix} = 0$

Self Practice Problems : 1. Show that $\vec{a}\cdot(\vec{b}+\vec{c})\times(\vec{a}+\vec{b}+\vec{c})=0$

2. One vertex of a parallelopiped is at the point A (1, –1, –2) in the rectangular cartesian co-ordinate. If three adjacent vertices are at B(–1, 0, 2), C(2, –2, 3) and D(4, 2, 1), then find the volume of the parallelopiped.
Ans. 72

3. Find the value of m such that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+m\hat{j}+5\hat{k}$ are coplanar.
Ans. – 4

4. Show that the vector $\vec{a}, \vec{b}, \vec{c}$, are coplanar if and only if $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$, $\vec{a}+\vec{b}$ are coplanar.

9. Vector Triple Product:

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be any three vectors, then the expression $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector & is called a vector triple product.

Geometrical Interpretation of $\vec{a} \times (\vec{b} \times \vec{c})$

Consider the expression $\vec{a} \times (\vec{b} \times \vec{c})$ which itself is a vector, since it is a cross product of two vectors $\vec{a}$ & $(\vec{b} \times \vec{c})$. Now $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector perpendicular to the plane containing $\vec{a}$ & $(\vec{b} \times \vec{c})$ but $\vec{b} \times \vec{c}$ is a vector perpendicular to the plane containing $\vec{b}$ & $\vec{c}$, therefore $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector which lies in the plane of $\vec{b}$ & $\vec{c}$ and perpendicular to $\vec{a}$. Hence we can express $\vec{a} \times (\vec{b} \times \vec{c})$ in terms of $\vec{b}$ & $\vec{c}$ i.e. $\vec{a} \times (\vec{b} \times \vec{c}) = x\vec{b} + y\vec{c}$ where x & y are scalars.

☞ $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ ☞ $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$

☞ $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$, in general

Solved Example For any vector $\vec{a}$, prove that $\hat{i} \times (\vec{a} \times \hat{i}) + \hat{j} \times (\vec{a} \times \hat{j}) + \hat{k} \times (\vec{a} \times \hat{k}) = 2\vec{a}$

Solution. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$. Then, $\hat{i} \times (\vec{a} \times \hat{i}) + \hat{j} \times (\hat{a} \times \hat{j}) + \hat{k} \times (\vec{a} \times \hat{k})$

$= \{(\hat{i} \cdot \hat{i})\vec{a} - (\hat{i} \cdot \vec{a})\hat{i}\} + \{(\hat{j} \cdot \hat{j})\vec{a} - (\hat{j} \cdot \vec{a})\hat{j}\} + \{(\hat{k} \cdot \hat{k})\vec{a} - (\hat{k} \cdot \vec{a})\hat{k}\}$

$= \{(\vec{a} - (\hat{i} \cdot \vec{a})\,\hat{i}\} + \{\vec{a} - (\hat{j} \cdot \vec{a})\hat{j}\} + \{\vec{a} - (\hat{k} \cdot \vec{a})\hat{k}\}$

$= 3\vec{a} - \{(\hat{i} \cdot \vec{a})\hat{i} + (\hat{j} \cdot \vec{a})\hat{j} + (\hat{k} \cdot \vec{a})\hat{k}$

$= 3\vec{a} - (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$

$= 3\vec{a} - \vec{a} = 2\vec{a}$

Solved Example Prove that $\vec{a} \times \{\vec{b} \times (\vec{c} \times \vec{a})\} = (\vec{b} \cdot \vec{d})(\vec{a} \times \vec{c}) - (\vec{b} \cdot \vec{c})\ (\vec{a} \times \vec{d})$

Solution. We have,

$\vec{a} \times \{\vec{b} \times (\vec{c} \times \vec{a})\} = \vec{a} \times \{(\vec{b} \cdot \vec{d})\,\vec{c} - (\vec{b} \cdot \vec{c})\,\vec{d}\}$

$= \vec{a} \times \{(\vec{b} \cdot \vec{d})\,\vec{c}\} - \vec{a} \times \{(\vec{b} \cdot \vec{c})\,\vec{d}\}$ [by dist. law]

$= (\vec{b} \cdot \vec{d})\,(\vec{a} \times \vec{c}) - (\vec{b} \cdot \vec{c}) - (\vec{b} \cdot \vec{c})\,(\vec{a} \times \vec{d})$

Solved Example: Let $\vec{a} = \alpha\hat{i} + 2\hat{j} - 3\hat{k}$, $\vec{b} = \hat{i} + 2\alpha\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \alpha\hat{j} + \hat{k}$. Find the value(s) of α, if any, such that $\{(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c})\} \times (\vec{c} \times \vec{a}) = 0$. Find the vector product when α = 0.

Solution.: $\{(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c})\} \times (\vec{c} \times \vec{a})$

$= [\vec{a}\ \vec{b}\ \vec{c}]\ \vec{b} \times (\vec{c} \times \vec{a})$ $= [\vec{a}\ \vec{b}\ \vec{c}]\ \{(\vec{a} \cdot \vec{b})\vec{c} - (\vec{b} \cdot \vec{c})\vec{a}\}$

which vanishes if (i) $(\vec{a} \cdot \vec{b})\ \vec{c} = (\vec{b} \cdot \vec{c})\ \vec{a}$ (ii) $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

(i) $(\vec{a} \cdot \vec{b})\ \vec{c} = (\vec{b} \cdot \vec{c})\ \vec{a}$ leads to the equation $2\alpha^3 + 10\alpha + 12 = 0$, $\alpha^2 + 6\alpha = 0$ and $6\alpha - 6 = 0$, which do not have a common solution. (ii) $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$$\Rightarrow \begin{vmatrix} \alpha & 2 & -3 \\ 1 & 2\alpha & -2 \\ 2 & -\alpha & 1 \end{vmatrix} = 0 \quad \Rightarrow \quad 3\alpha = 2 \quad \Rightarrow \quad \alpha = \frac{2}{3}$$

when α = 0, $[\vec{a}\ \vec{b}\ \vec{c}] = -10$, $\vec{a} \cdot \vec{b} = 6$, $\vec{b} \cdot \vec{c} = 0$ and the vector product is $-60\ (2\hat{i} + \hat{k})$.

Sol Ex If $\vec{A} + \vec{B} = \vec{a}$, $\vec{A} \cdot \vec{a} = 1$ and $\vec{A} \times \vec{B} = \vec{b}$, then prove that $\vec{A} = \dfrac{\vec{a} \times \vec{b} + \vec{a}}{|\vec{a}|^2}$ and $\vec{B} = \dfrac{\vec{b} \times \vec{a} + \vec{a}\,(|\vec{a}|^2 - 1)}{|\vec{a}|^2}$.

Solution.: Given $\vec{A} + \vec{B} = \vec{a}$ .....(i)

$\Rightarrow \vec{a} \cdot (\vec{A} + \vec{B}) = \vec{a} \cdot \vec{a}$ $\Rightarrow \vec{a} \cdot \vec{A} + \vec{a} \cdot \vec{B} = \vec{a} \cdot \vec{a}$ $\Rightarrow 1 + \vec{a} \cdot \vec{B} = |\vec{a}|^2$

$\Rightarrow \vec{a} \cdot \vec{B} = |\vec{a}|^2 - 1$ Given $\vec{A} \times \vec{B} = \vec{b} \Rightarrow$ $\vec{a} \times (\vec{A} \times \vec{B}) = \vec{a} \times \vec{b}$

$\Rightarrow (\vec{a} \cdot \vec{B})\ \vec{A} - (\vec{a} \cdot \vec{A})\ \vec{B} = \vec{a} \times \vec{b}$

$\Rightarrow (|\vec{a}|^2 - 1)\ \vec{A} - \vec{B} = \vec{a} \times \vec{b}$ [using equation (2)]

solving equation (1) and (5), simultaneously, we get

$$\vec{A} = \frac{\vec{a} \times \vec{b} + \vec{a}}{|\vec{a}|^2} \text{ and } \vec{B} = \frac{\vec{b} \times \vec{a} + a\,(|\vec{a}|^2 - 1)}{|\vec{a}|^2}$$

Sol. Ex. Solve for $\vec r$, the simultaneous equations $\vec r\times\vec b=\vec c\times\vec b$, $\vec r.\vec a=0$ provided $\vec a$ is not perpnedicular to $\vec b$.

Solution $(\vec r-\vec c)\times\vec b=0 \quad\Rightarrow\quad \vec r-\vec c$ and $\vec b$ are collinear

$\therefore\quad \vec r-\vec c=k\vec b \quad\Rightarrow\quad r=\vec c+k\vec b$ ....(i)

$\vec r.\vec a=0 \quad\Rightarrow\quad (\vec c+k\vec b).\vec a=0$

$\Rightarrow\quad k=-\dfrac{\vec a.\vec c}{\vec a.\vec b}$ putting in (i) we get $\vec r=\vec c-\dfrac{\vec a.\vec c}{\vec a.\vec b}\vec b$

Solved Example :If $\vec x\times\vec a+k\vec x=\vec b$, where k is a scalar and $\vec a,\vec b$ are any two vectors, then determine $\vec x$ in terms of $\vec a,\vec b$ and k.

Solution: $\vec x\times\vec a+k\vec x=\vec b$ ....(i)

Premultiple the given equation vectorially by $\vec a$

$\vec a\times(\vec x\times\vec a)+k(\vec a\times\vec x)=\vec a\times\vec b$

$\Rightarrow\quad (\vec a.\vec a)\vec x-(\vec a.\vec x)\vec a+k(\vec a\times\vec x)=\vec a\times\vec b$ ....(ii)

Premultiply (i) scalarly by $\vec a$

$[\vec a\ x\ \vec a]+k(\vec a.\vec x)=\vec a.\vec b$

$k(\vec a.\vec x)=\vec a.\vec b$ ....(iii)

Substituting $\vec x\times\vec a$ from (i) and $\vec a.\vec x$ from (iii) in (ii) we get

$$\vec x=\frac{1}{a^2+k^2}\left[k\vec b+(\vec a\times\vec b)+\frac{(\vec a.\vec b)}{k}\vec a\right]$$

Self Practice Problems :

1. Prove that $\vec a\times(\vec b\times\vec c)+\vec b\times(\vec c\times\vec a)+\vec c\times(\vec a\times\vec b)=0$.

2. Find the unit vector coplanar with $\hat i+\hat j+2\hat k$ and $\hat i+2\hat j+\hat k$ and perpendicular to $\hat i+\hat j+\hat k$.

Ans. $\dfrac{1}{\sqrt2}(-\hat j+\hat k)$ or, $\dfrac{1}{\sqrt2}(\hat j-\hat k)$

3. Prove that $\vec a\times\{\vec a\times(\vec a\times\vec b)\}=(\vec a.\vec a)(\vec b\times\vec a)$.

4. Given that $\vec x+\dfrac{1}{\vec p^2}(\vec p.\vec x)\vec p=\vec q$, show that $\vec p.\vec x=\dfrac12\vec p.\vec q$ and find $\vec x$ in terms of $\vec p$ and $\vec q$.

5. If $\vec x.\vec a=0$, $\vec x.\vec b=0$ and $\vec x.\vec c=0$ for some non-zero vector $\vec x$, then show that $[\vec a\,\vec b\,\vec c]=0$

6. Prove that $\vec r=\dfrac{(\vec r.\vec a)(\vec b\times\vec c)}{[abc]}+\dfrac{(\vec r.\vec b)(\vec c\times\vec a)}{[abc]}+\dfrac{(\vec r.\vec c)(\vec a\times\vec b)}{[abc]}$ where $\vec a,\vec b,\vec c$ are three non-coplanar vectors

10. Reciprocal System Of Vectors:

If $\vec a,\vec b,\vec c$ & $\vec a',\vec b',\vec c'$ are two sets of non coplanar vectors such that $\vec a.\vec a'=\vec b.\vec b'=\vec c.\vec c'=1$ then the two systems are called Reciprocal System of vectors.

Note: $\vec a=\dfrac{\vec b\times\vec c}{[\vec a\,\vec b\,\vec c]}\ \vec b=\dfrac{\vec c\times\vec a}{[\vec a\,\vec b\,\vec c]}\ \vec c=\dfrac{\vec a\times\vec b}{[\vec a\,\vec b\,\vec c]}$

Solved Example If $\vec a\,\vec b\,\vec c$ and $\vec a',\vec b',\vec c'$ be the reciprocal system of vectors, prove that

(i) $\vec a.\vec a'+\vec b.\vec b'+\vec c.\vec c'=3$ (ii) $\vec a\times\vec a'+\vec b\times\vec b'+\vec c\times\vec c'=\vec 0$

Solution. (i) We have : $\vec a.\vec a'=\vec b.\vec b'=\vec c.\vec c'=1$

$\vec a.\vec a'+\vec b.\vec b'+\vec c.\vec c'=1+1+1=3$

(ii) We have : $\vec a'=\lambda(\vec b\times\vec c)$, $\vec b'=\lambda(\vec c\times\vec a)$ and $\vec c'=\lambda((\vec a\times\vec b)$, where $\lambda=\dfrac{1}{[\vec a\,\vec b\,\vec c]}$

$\vec a\times\vec a'=\vec a\times\lambda(\vec b\times\vec c)=\lambda\{\vec a\times(\vec b\times\vec c)\}=\lambda\{(\vec a.\vec c)\vec b-(\vec a.\vec b)\vec c\}$

$\vec b\times\vec b'=\vec b\times\lambda(\vec c\times\vec a)=\lambda\{\vec b\times(\vec c\times\vec a)\}=\lambda\{(\vec b.\vec a)\vec c-(\vec b.\vec c)\vec a\}$

and $\vec c\times\vec c'=\vec c\times\lambda(\vec a\times\vec b)=\lambda\{\vec c\times(\vec a\times\vec b)\}=\lambda\{(\vec c.\vec b)\vec a-(\vec c.\vec a)\vec b\}$

$\therefore\quad \vec a\times\vec a'+\vec b\times\vec b'+\vec c\times\vec c'$

$=\lambda\{(\vec a.\vec c)\vec b-(\vec a.\vec b)\vec c\}+\lambda\{(\vec b.\vec a)\vec c-(\vec b.\vec c)\vec a\}+\lambda\{(\vec c.\vec b)\vec a-(\vec c.\vec a)\vec b\}$

$=\lambda[(\vec a.\vec c)\vec b-(\vec a.\vec b)\vec c+(\vec b.\vec a)\vec c-(\vec b.\vec c)\vec a+(\vec c.\vec b)\vec a-(\vec c.\vec a)\vec b]$

$=\lambda[(\vec a.\vec c)\vec b-(\vec a.\vec b)\vec c+(\vec a.\vec b)\vec c-(\vec b.\vec c)\vec a+(\vec b.\vec c)\vec a-(\vec a.\vec c)\vec b]$

$=\lambda\vec0=\vec0$